

CAMPUS HANDBOOK

BIEKE MASSELIS AND IVO DE PAUW

Animation Maths



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Content

<i>Acknowledgments</i>	15
Chapter 1 · Arithmetic Refresher	<u>17</u>
1.1 <i>Algebra</i>	18
Real Numbers	18
Real Polynomials	23
1.2 <i>Equations in one variable</i>	25
Linear Equations	25
Quadratic Equations	26
1.3 <i>Exercises</i>	32
Chapter 2 · Linear systems	<u>35</u>
2.1 <i>Definitions</i>	36
2.2 <i>Methods for solving linear systems</i>	38
Solving by substitution	38
Solving by elimination	39
2.3 <i>Exercises</i>	43
Chapter 3 · Trigonometry	<u>45</u>
3.1 <i>Angles</i>	46
3.2 <i>Triangles</i>	48
3.3 <i>Right Triangle</i>	52
3.4 <i>Unit Circle</i>	53
3.5 <i>Special Angles</i>	55
Trigonometric ratios for an angle of $45^\circ = \frac{\pi}{4}$ rad	56
Trigonometric ratios for an angle of $30^\circ = \frac{\pi}{6}$ rad	56
Trigonometric ratios for an angle of $60^\circ = \frac{\pi}{3}$ rad	57
Overview	57
3.6 <i>Pairs of Angles</i>	58
3.7 <i>Sum Identities</i>	58
3.8 <i>Inverse trigonometric functions</i>	61
3.9 <i>Exercises</i>	63

Chapter 4 · Functions	<u>65</u>
4.1 <i>Basic concepts on real functions</i>	66
4.2 <i>Polynomial functions</i>	67
Linear functions	67
Quadratic functions	69
4.3 <i>Intersecting functions</i>	71
4.4 <i>Trigonometric functions</i>	73
Elementary sine function	73
General sine function	73
Transversal oscillations	77
4.5 <i>Inverse trigonometric functions</i>	77
4.6 <i>Exercises</i>	80
Chapter 5 · The Golden Section	<u>83</u>
5.1 <i>The Golden Number</i>	84
5.2 <i>The Golden Section</i>	86
The Golden Triangle	86
The Golden Rectangle	87
The Golden Spiral	88
The Golden Pentagon	90
The Golden Ellipse	90
5.3 <i>Golden arithmetics</i>	91
Golden Identities	91
The Fibonacci Numbers	92
5.4 <i>The Golden Section worldwide</i>	94
5.5 <i>Exercises</i>	97
Chapter 6 · Coordinate systems	<u>99</u>
6.1 <i>Cartesian coordinates</i>	100
6.2 <i>Parametric curves</i>	100
6.3 <i>Polar coordinates</i>	103
6.4 <i>Polar curves</i>	106
A polar superformula	107
6.5 <i>Exercises</i>	109

Chapter 7 · Vectors	<u>111</u>
7.1 <i>The concept of a vector</i>	112
Vectors as arrows	112
Vectors as arrays	113
Free Vectors	116
Base Vectors	116
7.2 <i>Addition of vectors</i>	117
Vectors as arrows	117
Vectors as arrays	117
Vector addition summarized	118
7.3 <i>Scalar multiplication of vectors</i>	119
Vectors as arrows	119
Vectors as arrays	119
Scalar multiplication summarized	120
Properties	120
7.4 <i>Vector subtraction</i>	121
Creating free vectors	121
Euler's method for trajectories	122
7.5 <i>Decomposition of vectors</i>	123
Decomposition of a plane vector	123
Base vectors defined	124
7.6 <i>Dot product</i>	124
Definition	124
Geometric interpretation	126
Orthogonality	128
7.7 <i>Cross product</i>	129
Definition	129
Geometric interpretation	132
Parallelism	133
7.8 <i>Normal vectors</i>	135
7.9 <i>Exercises</i>	137
Chapter 8 · Parameters	<u>139</u>
8.1 <i>Parametric equations</i>	140
8.2 <i>Vector equation of a line</i>	141
8.3 <i>Intersecting straight lines</i>	145
8.4 <i>Vector equation of a plane</i>	147
8.5 <i>Exercises</i>	151

Chapter 9 · Kinematics	<u>153</u>
9.1 <i>Measures</i>	154
Precision	154
Units	154
9.2 <i>Deltatime</i>	155
9.3 <i>Translational motion</i>	155
Rectilinear motion with constant velocity (RMCV)	158
Rectilinear motion with constant acceleration (RMCA)	158
Free Fall	161
Summary	164
9.4 <i>Circular motion</i>	166
Uniform circular motion (UCM)	166
Nonuniform circular motion (NCM)	173
Summary	176
9.5 <i>Planar Curvilinear Motion</i>	177
Normal-tangential components	178
Radial-angular components	181
9.6 <i>Independence of Motion</i>	184
Combined rectilinear motions with constant velocity	184
Projectile motion (PM)	185
9.7 <i>Exercises</i>	190
Chapter 10 · Collision detection	<u>193</u>
10.1 <i>Collision detection using circles and spheres</i>	194
Circles and spheres	194
Intersecting line and circle	196
Intersecting circles and spheres	198
10.2 <i>Collision detection using vectors</i>	201
Location of a point with respect to other points	201
Altitude to a straight line	202
Altitude to a plane	204
Frame rate issues	206
Location of a point with respect to a polygon	207
10.3 <i>Exercises</i>	210

Chapter 11 · Matrices	<u>213</u>
11.1 <i>The concept of a matrix</i>	214
11.2 <i>Determinant of a square matrix</i>	215
11.3 <i>Addition of matrices</i>	217
11.4 <i>Scalar multiplication of a matrix</i>	219
11.5 <i>Transpose of a matrix</i>	220
11.6 <i>Dot product of matrices</i>	220
Introduction	220
Condition	222
Definition	222
Properties	223
11.7 <i>Inverse of a matrix</i>	225
Introduction	225
Definition	225
Conditions	226
Row reduction	226
Matrix inversion	227
Inverse of a product	230
Solving systems of linear equations	231
11.8 <i>The Fibonacci operator</i>	233
11.9 <i>Exercises</i>	235
 Chapter 12 · Linear transformations	 <u>237</u>
12.1 <i>Translation</i>	238
12.2 <i>Scaling</i>	243
12.3 <i>Rotation</i>	246
Rotation in 2D	246
Rotation in 3D	248
12.4 <i>Reflection</i>	250
12.5 <i>Shearing</i>	251
12.6 <i>Composing basic transformations</i>	254
2D rotation around an arbitrary center	256
3D scaling about an arbitrary center	259
2D reflection over an axis through the origin	260
2D reflection over an arbitrary axis	261
3D combined rotation	264
12.7 <i>Conventions</i>	265
12.8 <i>Exercises</i>	266

Chapter 13 · Hypercomplex numbers	<u>269</u>
13.1 <i>Complex numbers</i>	270
13.2 <i>Complex number arithmetics</i>	273
Complex conjugate	273
Addition and subtraction	274
Multiplication	275
Division	277
13.3 <i>Complex numbers and transformations</i>	279
13.4 <i>Complex continuation of the Fibonacci numbers</i>	281
Integer Fibonacci numbers	281
Complex Fibonacci numbers	282
13.5 <i>Quaternions</i>	283
13.6 <i>Quaternion arithmetics</i>	284
Addition and subtraction	285
Multiplication	285
Quaternion conjugate	287
Inverse quaternion	288
13.7 <i>Quaternions and rotation</i>	288
13.8 <i>Exercises</i>	293
Chapter 14 · Bezier curves	<u>295</u>
14.1 <i>Vector equation of segments</i>	296
Linear Bezier segment	296
Quadratic Bezier segment	297
Cubic Bezier segment	298
Bezier segments of higher degree	300
14.2 <i>De Casteljau algorithm</i>	301
14.3 <i>Bezier curves</i>	302
Concatenation	302
Linear transformations	304
Illustrations	304
14.4 <i>Matrix representation</i>	306
Linear Bezier segment	306
Quadratic Bezier segment	307
Cubic Bezier segment	308
14.5 <i>B-splines</i>	310
Cubic B-splines	310
Matrix representation	311
De Boor's algorithm	313
14.6 <i>Exercises</i>	315

Annex A · Real numbers in computers	<u>317</u>
A.1 <i>Scientific notation</i>	317
A.2 <i>The decimal computer</i>	317
A.3 <i>Special values</i>	318
Annex B · Notations and Conventions	<u>319</u>
B.1 <i>Alphabets</i>	319
Latin alphabet	319
Greek alphabet	319
B.2 <i>Mathematical symbols</i>	320
Sets	320
Mathematical symbols	321
Mathematical keywords	321
Numbers	322
Annex C · The International System of Units (SI)	<u>323</u>
C.1 <i>SI Prefixes</i>	323
C.2 <i>SI Base measures</i>	324
C.3 <i>SI Supplementary measure</i>	324
C.4 <i>SI Derived measures</i>	325
Annex D · Companion website	<u>327</u>
D.1 <i>Interactivities</i>	327
D.2 <i>Solutions</i>	327
<i>Bibliography</i>	328
<i>Index</i>	331

This book is dedicated to Malaika.

*“Sometimes I’m black, sometimes I’m white
it all depends on who is on the other side
there are things they can not see
and there are things I can not hide”*

Bruno Deneckere (Someday, June 2006)

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Chapter 1 · Arithmetic Refresher



As this chapter offers all necessary mathematical skills for a full mastering of all further topics explained in this book, we strongly recommend it. To serve its purpose, the successive paragraphs below refresh some required aspects of mathematical language as used on the applied level.

1.1 Algebra

REAL NUMBERS

We typeset the set of:

- ▷ natural numbers (unsigned integers) as \mathbb{N} including zero,
- ▷ integer numbers as \mathbb{Z} including zero,
- ▷ rational numbers as \mathbb{Q} including zero,
- ▷ real numbers (floats) as \mathbb{R} including zero.

All the above make a chain of subsets: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

To avoid possible confusion, we outline a brief glossary of mathematical terms. We recall that using the correct mathematical terms reflects a correct mathematical thinking. Putting down ideas in the correct words is of major importance for a profound insight.

Sets

- ▷ We recall writing all **subsets** in between braces, e.g. the **empty set** appears as $\{\}$.
- ▷ We define a **singleton** as any subset containing only one element, e.g. $\{5\} \subset \mathbb{N}$, as a subset of natural numbers.
- ▷ We define a **pair** as any subset containing just two elements, e.g. $\{115, -4\} \subset \mathbb{Z}$, as a subset of integers. In programming the boolean values *true* and *false* make up a pair $\{true, false\}$ called the boolean set which we typeset as \mathbb{B} .
- ▷ We define $\mathbb{Z}^- = \{\dots, -3, -2, -1\}$ whenever we need negative integers only. We express symbolically that -1234 is an **element** of \mathbb{Z}^- by typesetting $-1234 \in \mathbb{Z}^-$.
- ▷ We typeset the **setminus** operator to delete elements from a set by using a backslash, e.g. $\mathbb{N} \setminus \{0\}$ reading all natural numbers except zero, $\mathbb{Q} \setminus \mathbb{Z}$ meaning all pure rational numbers after all integer values left out and $\mathbb{R} \setminus \{0, 1\}$ expressing all real numbers apart from zero and one.

Calculation basics

operation	example	a	b	c
to add	$a + b = c$	term	term	sum
to subtract	$a - b = c$	term	term	difference
to multiply	$a \cdot b = c$	factor	factor	product
to divide	$\frac{a}{b} = c, b \neq 0$	numerator	divisor or denominator	quotient or fraction
to exponentiate	$a^b = c$	base	exponent	power
to take root	$\sqrt[b]{a} = c$	radicand	index	radical

We write the **opposite** of a real number r as $-r$, defined by the sum $r + (-r) = 0$. We typeset the **reciprocal** of a nonzero real number r as $\frac{1}{r}$ or r^{-1} , defined by the product $r \cdot r^{-1} = 1$.

We define **subtraction** as equivalent to adding the opposite: $a - b = a + (-b)$. We define **division** as equivalent to multiplying with the reciprocal: $a : b = a \cdot b^{-1}$.

When we mix operations we need to apply priority rules for them. There is a fixed priority list ‘PEMDAS’ in performing mixed operations in \mathbb{R} that can easily be memorized by ‘Please Excuse My Dear Aunt Sally’.

- ▷ First process all that is delimited in between Parentheses,
- ▷ then Exponentiate,
- ▷ then Multiply and Divide from left to right,
- ▷ finally Add and Subtract from left to right.

Now we discuss the **distributive law** ruling within \mathbb{R} , which we define as threading a ‘superior’ operation over an ‘inferior’ operation. Conclusively, distributing requires two *different* operations.

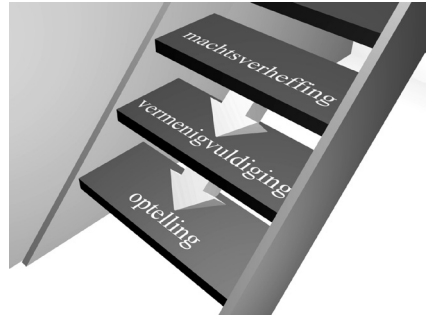
Hence we distribute *exponentiating* over *multiplication* as in $(a \cdot b)^3 = a^3 \cdot b^3$. Likewise rules *multiplying* over *addition* as in $3 \cdot (a + b) = 3 \cdot a + 3 \cdot b$.

However we should never stumble on this ‘Stairway of Distributivity’ by going too fast:

$$(a + b)^3 \neq a^3 + b^3,$$

$$\sqrt{a + b} \neq \sqrt{a} + \sqrt{b},$$

$$\sqrt{x^2 + y^2} \neq x + y.$$



Fractions

A **fraction** is what we call any rational number written as $\frac{t}{n}$ given $t, n \in \mathbb{Z}$ and $n \neq 0$, wherein t is called the **numerator** and n the **denominator**. We define the reciprocal of a nonzero fraction $\frac{t}{n}$ as $\frac{1}{\frac{t}{n}} = \frac{n}{t}$ or as the power $\left(\frac{t}{n}\right)^{-1}$. We define the opposite fraction as $-\frac{t}{n} = \frac{-t}{n} = \frac{t}{-n}$. We summarize fractional arithmetics:

$$\text{sum} \quad \frac{t}{n} + \frac{a}{b} = \frac{t \cdot b + n \cdot a}{n \cdot b},$$

$$\text{difference} \quad \frac{t}{n} - \frac{a}{b} = \frac{t \cdot b - n \cdot a}{n \cdot b},$$

$$\text{product} \quad \frac{t}{n} \cdot \frac{a}{b} = \frac{t \cdot a}{n \cdot b},$$

$$\text{division} \quad \frac{\frac{t}{n}}{\frac{a}{b}} = \frac{t}{n} \cdot \frac{b}{a},$$

$$\text{exponentiation} \quad \left(\frac{t}{n}\right)^m = \frac{t^m}{n^m},$$

$$\text{singular fractions} \quad \frac{1}{0} = \pm\infty \text{ infinity,}$$

$$\frac{0}{0} = ? \text{ indeterminate.}$$

Powers

We define a **power** as any real number written as g^m , wherein g is called its **base** and m its **exponent**. The opposite of g^m is simply $-g^m$. The reciprocal of g^m is $\frac{1}{g^m} = g^{-m}$, given $g \neq 0$.

According to the exponent type we distinguish between:

$$\begin{array}{ll}
 g^3 = g \cdot g \cdot g & 3 \in \mathbb{N}, \\
 g^{-3} = \frac{1}{g^3} = \frac{1}{g \cdot g \cdot g} & -3 \in \mathbb{Z}, \\
 g^{\frac{1}{3}} = \sqrt[3]{g} = w \Leftrightarrow w^3 = g & \frac{1}{3} \in \mathbb{Q}, \\
 g^0 = 1 & g \neq 0.
 \end{array}$$

Whilst calculating powers we may have to:

$$\begin{array}{ll}
 \text{multiply} & g^3 \cdot g^2 = g^{3+2} = g^5, \\
 \text{divide} & \frac{g^3}{g^2} = g^3 \cdot g^{-2} = g^{3-2} = g^1, \\
 \text{exponentiate} & (g^3)^2 = g^{3 \cdot 2} = g^6 \text{ them.}
 \end{array}$$

We insist on avoiding typesetting radicals like $\sqrt[7]{g^3}$ and strongly recommend their contemporary notation using radicand g and exponent $\frac{3}{7}$, consequently exponentiating g to $g^{\frac{3}{7}}$. We recall the fact that all square roots are non-negative numbers, $\sqrt{a} = a^{\frac{1}{2}} \in \mathbb{R}^+$ for $a \in \mathbb{R}^+$.

As well knowing the above exponent types as understanding the above rules to calculate them are inevitable to use powers successfully. We advise memorizing the integer squares running from $1^2 = 1$, $2^2 = 4$, ..., up to $15^2 = 225$, $16^2 = 256$ and the integer cubes running from $1^3 = 1$, $2^3 = 8$, ..., up to $7^3 = 343$, $8^3 = 512$ in order to easily recognize them.

Recall that the only way out of any power is exponentiating with its reciprocal exponent. For this purpose we need to exponentiate both left hand side and right hand side of any given relation (see also paragraph 1.2).

Example: Find x when $\sqrt[7]{x^3} = 5$ by exponentiating this power.

$$x^{\frac{3}{7}} = 5 \iff \left(x^{\frac{3}{7}}\right)^{\frac{7}{3}} = (5)^{\frac{7}{3}} \iff x \approx 42.7494.$$

We emphasize the above strategy as the only successful one to free base x from its exponent, yielding its correct expression numerically approximated if we like to.

Example: Find x when $x^2 = 5$ by exponentiating this power.

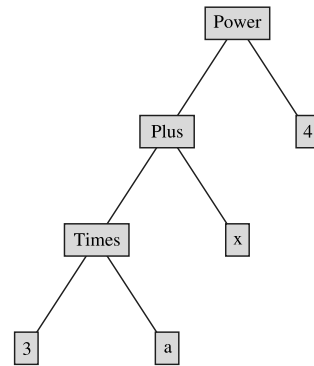
$$x^2 = 5 \iff (x^2)^{\frac{1}{2}} = (5)^{\frac{1}{2}} \text{ or } -(5)^{\frac{1}{2}} \iff x \approx 2.23607 \text{ or } -2.23607.$$

We recall the above double solution whenever we free base x from an *even* exponent, yielding their correct expression as accurate as we like to.

Mathematical expressions

Composed mathematical expressions can often seem intimidating or cause confusion. To gain transparency in them, we firstly recall indexed variables which we define as subscripted to count them: $x_1, x_2, x_3, x_4, \dots, x_{99999}, x_{100000}, \dots$, and $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \dots$. It is common practice in industrial research to use thousands of variables, so just picking unindexed characters would be insufficient. Taking our own alphabet as an example, it would only provide us with 26 characters.

We define finite expressions as composed of (mathematical) operations on objects (numbers, variables or structures). We can for instance analyze the expression $(3a+x)^4$ by drawing its **tree form**. This example reveals a Power having exponent 4 and a subexpression in its base. The base itself yields a sum of the variable x Plus another subexpression. This final subexpression shows the product 3 Times a .



Let us also evaluate this expression $(3a+x)^4$. Say $a = 1$, then we see our expression partly collaps to $(3+x)^4$. If we on top of this assign $x = 2$, our expression then finally turns to the numerical value $(3+2)^4 = 5^4 = 625$.

When we expand this power to its **pure sum expression** $81a^4 + 108a^3x + 54a^2x^2 + 12ax^3 + x^4$, we did nothing but *reshape* its **pure product expression** $(3a+x)^4$.

We warn that trying to solve this expression - which is not a relation - is completely in vain. Recall that inequalities, equations and systems of equations or inequalities are the only objects in the universe we can (try to) solve mathematically.

Relational operators

We also refresh the use of correct terms for inequalities and equations.

We define an **inequality** as any *variable* expression comparing a left hand side to a right hand side by applying the ‘is-(strictly)-less-than’ or by applying the ‘is-(strictly)-greater-than’ operator. For example, we can read $(3a+x)^4 \leq (b+4)(x+3)$ containing variables a, x, b . Consequently we may solve such inequality for any of the unknown quantities a, x or b .

We define an **equation** as any *variable* expression comparing a left hand side to a right hand side by applying the ‘is-equal-to’ operator. For example $(3a+x)^4 = (b+4)(x+3)$ is an equation containing variables a, x, b . Consequently we also may solve equations for