CAMPUS HANDBOOK

BIEKE MASSELIS AND IVO DE PAUW

Animation Maths

LANNOO CAMPUS

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Content

This book is dedicated to Malaika.

“Sometimes I’m black, sometimes I’m white
it all depends on who is on the other side
there are things they can not see
and there are things I can not hide”

Bruno Deneckere (Someday, June 2006)

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Chapter 1 · Arithmetic Refresher

As this chapter offers all necessary mathematical skills for a full mastering of all further topics explained in this book, we strongly recommend it. To serve its purpose, the successive paragraphs below refresh some required aspects of mathematical language as used on the applied level.

1.1 Algebra

Real Numbers

We typeset the set of:

- ▷ natural numbers (unsigned integers) as $\mathbb{N}$ including zero,
- ▷ integer numbers as $\mathbb{Z}$ including zero,
- ▷ rational numbers as $\mathbb{Q}$ including zero,
- ▷ real numbers (floats) as $\mathbb{R}$ including zero.

All the above make a chain of subsets: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

To avoid possible confusion, we outline a brief glossary of mathematical terms. We recall that using the correct mathematical terms reflects a correct mathematical thinking. Putting down ideas in the correct words is of major importance for a profound insight.

Sets

- ▷ We recall writing all **subsets** in between braces, e.g. the **empty set** appears as $\{\}$.
- ▷ We define a **singleton** as any subset containing only one element, e.g. $\{5\} \subset \mathbb{N}$, as a subset of natural numbers.
- ▷ We define a **pair** as any subset containing just two elements, e.g. $\{115, -4\} \subset \mathbb{Z}$, as a subset of integers. In programming the boolean values *true* and *false* make up a pair $\{true, false\}$ called the boolean set which we typeset as $\mathbb{B}$.
- ▷ We define $\mathbb{Z}^- = \{\ldots, -3, -2, -1\}$ whenever we need negative integers only. We express symbolically that -1234 is an **element** of $\mathbb{Z}^-$ by typesetting $-1234 \in \mathbb{Z}^-$.
- ▷ We typeset the **setminus** operator to delete elements from a set by using a backslash, e.g. $\mathbb{N} \setminus \{0\}$ reading all natural numbers except zero, $\mathbb{Q} \setminus \mathbb{Z}$ meaning all pure rational numbers after all integer values left out and $\mathbb{R} \setminus \{0, 1\}$ expressing all real numbers apart from zero and one.

Calculation basics

operation	**example**	***a***	***b***	***c***
to add	$a+b=c$	term	term	sum
to subtract	$a-b=c$	term	term	difference
to multiply	$a \cdot b = c$	factor	factor	product
to divide	$\frac{a}{b}=c, b \neq 0$	numerator	divisor or denominator	quotient or fraction
to exponentiate	$a^b = c$	base	exponent	power
to take root	$\sqrt[b]{a} = c$	radicand	index	radical

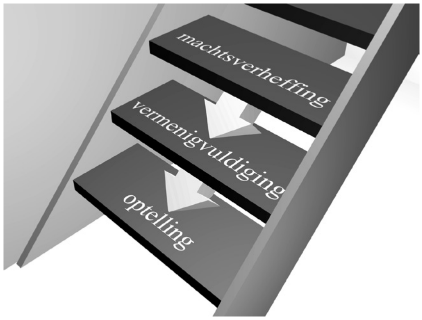
We write the **opposite** of a real number r as $-r$, defined by the sum $r+(-r)=0$. We typeset the **reciprocal** of a nonzero real number r as $\frac{1}{r}$ or r^{-1}, defined by the product $r \cdot r^{-1} = 1$.

We define **subtraction** as equivalent to adding the opposite: $a-b=a+(-b)$. We define **division** as equivalent to multiplying with the reciprocal: $a:b=a \cdot b^{-1}$.

When we mix operations we need to apply priority rules for them. There is a fixed priority list 'PEMDAS' in performing mixed operations in $\mathbb{R}$ that can easily be memorized by 'Please Excuse My Dear Aunt Sally'.

- ▷ First process all that is delimited in between Parentheses,
- ▷ then Exponentiate,
- ▷ then Multiply and Divide from left to right,
- ▷ finally Add and Subtract from left to right.

Now we discuss the **distributive law** ruling within $\mathbb{R}$, which we define as threading a 'superior' operation over an 'inferior' operation. Conclusively, distributing requires two *different* operations.



Hence we distribute *exponentiating* over *multiplication* as in $(a \cdot b)^3 = a^3 \cdot b^3$. Likewise rules *multiplying* over *addition* as in $3 \cdot (a+b) = 3 \cdot a + 3 \cdot b$.

However we should never stumble on this 'Stairway of Distributivity' by going too fast:

$$(a+b)^3 \neq a^3 + b^3,$$

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b},$$

$$\sqrt{x^2+y^2} \neq x+y.$$

Fractions

A **fraction** is what we call any rational number written as $\frac{t}{n}$ given $t, n \in \mathbb{Z}$ and $n \neq 0$, wherein t is called the **numerator** and n the **denominator**. We define the reciprocal of a nonzero fraction $\frac{t}{n}$ as $\frac{1}{\frac{t}{n}} = \frac{n}{t}$ or as the power $\left(\frac{t}{n}\right)^{-1}$. We define the opposite fraction as $-\frac{t}{n} = \frac{-t}{n} = \frac{t}{-n}$. We summarize fractional arithmetics:

sum	$\frac{t}{n} + \frac{a}{b} = \frac{t \cdot b + n \cdot a}{n \cdot b}$,
difference	$\frac{t}{n} - \frac{a}{b} = \frac{t \cdot b - n \cdot a}{n \cdot b}$,
product	$\frac{t}{n} \cdot \frac{a}{b} = \frac{t \cdot a}{n \cdot b}$,
division	$\frac{\frac{t}{n}}{\frac{a}{b}} = \frac{t}{n} \cdot \frac{b}{a}$,
exponentiation	$\left(\frac{t}{n}\right)^m = \frac{t^m}{n^m}$,
singular fractions	$\frac{1}{0} = \pm\infty$ infinity,
	$\frac{0}{0} =?$ indeterminate.

Powers

We define a **power** as any real number written as g^m , wherein g is called its **base** and m its **exponent**. The opposite of g^m is simply $-g^m$. The reciprocal of g^m is $\frac{1}{g^m} = g^{-m}$, given $g \neq 0$.

According to the exponent type we distinguish between:

$$
\begin{aligned}
&g^3 = g\cdot g\cdot g && 3\in\mathbb{N},\\
&g^{-3} = \tfrac{1}{g^3} = \tfrac{1}{g\cdot g\cdot g} && -3\in\mathbb{Z},\\
&g^{\frac{1}{3}} = \sqrt[3]{g} = w \Leftrightarrow w^3 = g && \tfrac{1}{3}\in\mathbb{Q},\\
&g^0 = 1 && g\neq 0.
\end{aligned}
$$

Whilst calculating powers we may have to:

$$
\begin{aligned}
&\text{multiply} && g^3\cdot g^2 = g^{3+2} = g^5,\\
&\text{divide} && \tfrac{g^3}{g^2} = g^3\cdot g^{-2} = g^{3-2} = g^1,\\
&\text{exponentiate} && \left(g^3\right)^2 = g^{3\cdot 2} = g^6 \ \text{ them.}
\end{aligned}
$$

We insist on avoiding typesetting radicals like $\sqrt[7]{g^3}$ and strongly recommend their contemporary notation using radicand g and exponent $\frac{3}{7}$, consequently exponentiating g to $g^{\frac{3}{7}}$. We recall the fact that all square roots are non-negative numbers, $\sqrt{a} = a^{\frac{1}{2}} \in \mathbb{R}^+$ for $a\in\mathbb{R}^+$.

As well knowing the above exponent types as understanding the above rules to calculate them are inevitable to use powers successfully. We advise memorizing the integer squares running from $1^2 = 1$, $2^2 = 4$, …, up to $15^2 = 225$, $16^2 = 256$ and the integer cubes running from $1^3 = 1$, $2^3 = 8$, …, up to $7^3 = 343$, $8^3 = 512$ in order to easily recognize them.

Recall that the only way out of any power is exponentiating with its reciprocal exponent. For this purpose we need to exponentiate both left hand side and right hand side of any given relation (see also paragraph 1.2).

Example: Find x when $\sqrt[7]{x^3} = 5$ by exponentiating this power.

$$x^{\frac{3}{7}} = 5 \Longleftrightarrow \left(x^{\frac{3}{7}}\right)^{\frac{7}{3}} = (5)^{\frac{7}{3}} \Longleftrightarrow x \approx 42.7494.$$

We emphasize the above strategy as the only successful one to free base x from its exponent, yielding its correct expression numerically approximated if we like to.

Example: Find x when $x^2 = 5$ by exponentiating this power.

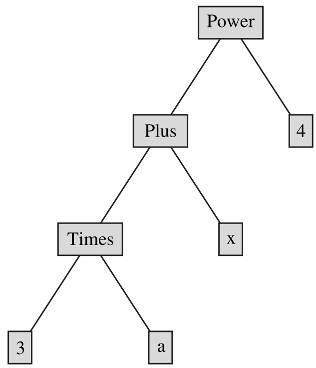
$$x^2 = 5 \Longleftrightarrow \left(x^2\right)^{\frac{1}{2}} = (5)^{\frac{1}{2}} \text{ or } -(5)^{\frac{1}{2}} \Longleftrightarrow x \approx 2.23607 \text{ or} -2.23607.$$

We recall the above double solution whenever we free base x from an *even* exponent, yielding their correct expression as accurate as we like to.

Mathematical expressions

Composed mathematical expressions can often seem intimidating or cause confusion. To gain transparancy in them, we firstly recall indexed variables which we define as subscripted to count them: $x_1, x_2, x_3, x_4, \ldots, x_{99999}, x_{100000}, \ldots$, and $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots$. It is common practice in industrial research to use thousands of variables, so just picking unindexed characters would be insufficient. Taking our own alphabet as an example, it would only provide us with 26 characters.

We define finite expressions as composed of (mathematical) operations on objects (numbers, variables or structures). We can for instance analyze the expression $(3a+x)^4$ by drawing its **tree form**. This example reveals a Power having exponent 4 and a subexpression in its base. The base itself yields a sum of the variable x Plus another subexpression. This final subexpression shows the product 3 Times a.



Let us also evaluate this expression $(3a+x)^4$. Say $a = 1$, then we see our expression partly collaps to $(3+x)^4$. If we on top of this assign $x = 2$, our expression then finally turns to the numerical value $(3+2)^4 = 5^4 = 625$.

When we expand this power to its **pure sum expression** $81a^4 + 108a^3x + 54a^2x^2 + 12ax^3 + x^4$, we did nothing but *reshape* its **pure product expression** $(3a+x)^4$.

We warn that trying to solve this expression - which is not a relation - is completely in vain. Recall that inequalities, equations and systems of equations or inequalities are the only objects in the universe we can (try to) solve mathematically.

Relational operators

We also refresh the use of correct terms for inequalities and equations.

We define an **inequality** as any *variable* expression comparing a left hand side to a right hand side by applying the 'is-(strictly)-less-than' or by applying the 'is-(strictly)-greater-than' operator. For example, we can read $(3a+x)^4 \leqslant (b+4)(x+3)$ containing variables *a, x, b*. Consequently we may solve such inequality for any of the unknown quantities a, x or b.

We define an **equation** as any *variable* expression comparing a left hand side to a right hand side by applying the 'is-equal-to' operator. For example $(3a+x)^4 = (b+4)(x+3)$ is an equation containing variables *a, x, b*. Consequently we also may solve equations for